# Lecture 5 Twisted int forms $\rightsquigarrow$ CG invts

Last time: int forms + linking forms

This time: incorporate more $\pi_1$- information.

en route: Alex module
destination: CG invts.

## Homology w. twisted coefficients

Let $R$ be a ring w. involution, denoted $\bar{\cdot}$. Assume $\overline{r \cdot s} = \bar{s} \cdot \bar{r}$

$M$ a left $R$-module then $\bar{M}$ the assoc right $R$-module given by $m \cdot r = \bar{r} m \quad \forall m \in M, r \in R$.

Let $X$ connected CW cx,
$Y \subseteq X$ subcx (poss. empty)

Let $\begin{array}{c}\tilde{X}\\ \downarrow p\\ X\end{array}$ the univ. cover and $\tilde{Y} := p^{-1}(Y)$.

Then $C_*(\tilde{X}, \tilde{Y})$ is a left $\mathbb{Z}[\pi_1 X]$-module via deck gp action.

$\boxed{\text{Input: } \varphi : \mathbb{Z}[\pi_1 X] \to R}$

Then $R$ is a $(R, \mathbb{Z}[\pi_1 X])$-bimodule

__Def^n:__ The homology of $(X,Y)$ __twisted by $\varphi$__, or __w. coeffs in $R$__ is given by
$$H_*^{\varphi}(X,Y;R) := H_*\left(R \otimes_{\mathbb{Z}[\pi_1 X]} C(\tilde{X}, \tilde{Y})\right)$$

Note: this is an $R$-module for each $*$.

Similarly, $H^*_{\varphi}(X,Y;R) := H_*\left(\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi_1 X]}\left(\overline{C_*(\tilde{X},\tilde{Y})}, R\right)\right)$

Warning: sometimes $\varphi$ dropped from notation.

E.g. __Alexander module__ $K \subseteq S^3$ a knot. $X(K) := S^3 \setminus \mathring{\nu} K$
$$R := \mathbb{Z}[t^{\pm 1}], \quad \varphi : \mathbb{Z}[\pi_1 X] \to \mathbb{Z}[t^{\pm 1}]$$
$$\mu_K \mapsto t$$
$\boxed{\text{oriented}}$

Then: $H_1^{\varphi}(X(K); \mathbb{Z}[t^{\pm 1}])$ is the Alex. module of $K$, denoted by $\mathcal{A}(K)$.

Alternatively, notice that since $\exists\ \pi_1(X(K)) \xrightarrow{ab} \mathbb{Z}$
we have an infinite cyclic cover $\begin{matrix}\widehat{X(K)}\\ \downarrow \\ X(K)\end{matrix}$
with a deck gp. action (induced) of $\mathbb{Z}[t^{\pm 1}]$ on $C_*(\widehat{X(K)})$
(see HW) Then $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$ is the hom. of $\widehat{X(K)}$

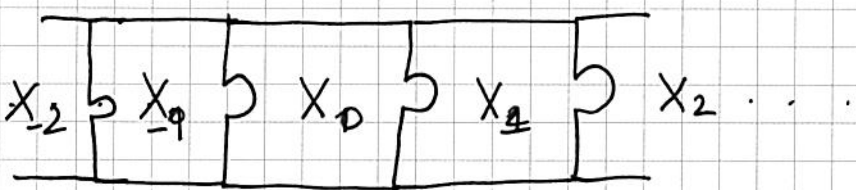
Note: $H_1(\widehat{X(K)}) \cong \pi_1(X(K))^{(1)} / \pi_1(X(K))^{(2)}$ considered as a $\mathbb{Z}[t^{\pm 1}]$ module
where $G^{(n)}$ for a gp $G$ is the $n$th term of the derived series.
i.e. $G^{(0)} := G$, $G^{(n+1)} := [G^{(n)}, G^{(n)}]$

So: $\mathcal{A}(K)$ trivial $\iff$ $\pi_1(X(K))^{(1)} = \pi_1(X(K))^{(2)}$
i.e. $[\pi_1(X(K)), \pi_1(X(K))]$ is perfect.

<u>Facts</u>: $\mathcal{A}(K)$ is a finitely generated, torsion $\mathbb{Z}[t^{\pm 1}]$-module $\forall K$
i.e. $\forall m \in \mathcal{A}(K)$, $\exists\ 0 \neq r \in \mathbb{Z}[t^{\pm 1}]$ s.t. $r.m = 0$.

In fact: the Alex. poly. $\Delta_t(K)$ annihilates the module
and if $V$ is a Seifert ~~surface~~ matrix for $K$, then $V - tV^T$ is a presentation matrix for $\mathcal{A}(K)$.

<u>Idea of proof</u>: (i) Build $\widehat{X(K)}$ by gluing together $\infty$ copies of $S^3 \setminus F \times [-1,1]$ for $F$ a Seifert surface




ii) Claim: $H_1(S^3 \setminus F \times (-1,1))$ gen. by "dual" curves $\{\alpha_1, \ldots, \alpha_{2g}\}$


"Alexander duality"

iii) To glue $X_i$ to $X_{i+1}$, must identify $\underset{\underset{V}{\updownarrow}}{a_i^+}$ with $\underset{\underset{tV^T}{\updownarrow}}{ta_i^-}$

[see Rolfsen 8.C.13]

Wish: Alex poly should be the order of $\mathcal{A}(K)$

Problem: what does "order" mean?

**Solutions**

1. Consider $\mathcal{A}^{\mathbb{Q}}(K) := H_1(X(K); \mathbb{Q}[t^{\pm 1}])$ via same recipe "rational Alex module"
   $\mathbb{Q}[t^{\pm1}]$ is a PID & $\Delta_t(K) = \text{order}_{\mathbb{Q}[t^{\pm1}]}(\mathcal{A}^{\mathbb{Q}}(K))$
2. ~~Re~~ Modules over $\mathbb{Z}[t^{\pm1}]$ have an "order ideal" for $\mathcal{A}(K)$, this is principal & $\Delta_t(K)$ is a generator.
   Note: generators only well-defined upto units in $\mathbb{Z}[t^{\pm1}]$ such as $\pm t^i$.

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$\mathcal{A}(K)$ is a torsion module. Is there a corr. linking form?

$$Bl : \mathcal{A}(K) \times \mathcal{A}(K) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]$$
$$x \qquad y \longmapsto \frac{1}{q(t)} \sum_{-\infty}^{\infty} (D \cdot t^i y)\, t^i$$

← note this exponent is different from lecture!

$\exists q(t) \neq 0$ s.t. $q(t)x = 0$

$\Rightarrow \exists D$ s.t. $\partial D = q(t)x$

The Blanchfield form on $\mathcal{A}(K)$

Facts: $Bl(p(t)x, y) = p(t)\, Bl(x,y)$
$Bl(x, p(t)y) = Bl(x,y)\, p(t^{-1})$ } "sesquilinear"
$Bl(x,y) = \overline{Bl(y,x)}$ replace all $t$'s by $t^{-1}$'s "Hermitian"
$Bl : \mathcal{A}(K) \xrightarrow{\cong} \overline{\text{Hom}_{\mathbb{Z}[t^{\pm1}] \text{(left)}}(\mathcal{A}(K), \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}])}$ ~~isom~~ "nonsingular"

Any form $\mathcal{B}$ over a torsion module $\mathcal{A}$ with above props is called a linking form

$(\mathcal{A}, \mathcal{B})$ is called hyperbolic (metabolic?) if $\exists P \subseteq \mathcal{A}$ submod s.t. $P = P^{\perp} := \{x \in \mathcal{A} \mid \mathcal{B}(x,y) = 0 \ \forall y \in P\}$

[Kearton] $K \subseteq S^3$ is algebraically slice iff $(\mathcal{A}(K), Bl(K))$ is hyperbolic.

Furthermore, there is a Witt group of linking forms

$$\left(\{\text{linking forms}\}/_{\text{hyp. forms}}, \oplus\right)$$

$\downarrow \cong$ (restricted to torsion modules, square pres. matrix)

$\mathcal{G}$ alg conc. gp

There are also hoisted intersection forms.

Same setup as for usual int forms, but now more data.

As before, $R$ ring w. involution, $\varphi: \mathbb{Z}[\pi_1 X] \to R$, $\varphi(g^{-1}) = \overline{\varphi(g)}$ $\forall g \in \pi_1 X$.

$X^n$ compact, oriented, connected $n$-mfld

$$\begin{array}{ll} \text{Then we have } Q_X^{\varphi}: H_p^{\varphi}(X;R) & \longrightarrow \overline{\mathrm{Hom}_{\text{left-}R}(H_{n-p}^{\varphi}(X;R),R)} \\ \quad\quad\quad\quad\quad\quad \downarrow & \quad\quad\quad\quad \uparrow \kappa \\ \quad\quad\quad\quad H_p^{\varphi}(X,\partial X;R) & \xrightarrow[\text{P.D.}]{} H_{\varphi}^{n-p}(X;R) \end{array}$$

"hoisted intersection form"

Application / example:

$X^4$ a 4-mfld, $\pi_1(X) \to \mathbb{Z}/m$ for $m \geq 1$

$\rightsquigarrow$ $m$-fold cyclic cover $X_m \downarrow X$ with $t: X_m \to X_m$ generator of deck gp.

so $H_2(X_m;\mathbb{Z})$ has a $\mathbb{Z}[\mathbb{Z}/m]$-mod structure

Let $\omega := \omega_m = e^{2\pi i/m}$

$\mathbb{Z}[\mathbb{Z}/m] \longrightarrow \mathbb{Q}(\omega)$, $\underset{\mathbb{Z}/m}{1} \longmapsto \omega_m$. Then $\mathbb{Q}(\omega)$ is a $(\mathbb{Q}(\omega), \mathbb{Z}[\mathbb{Z}/m])$-bimodule

$\varphi: \pi_1(X) \to \mathbb{Z}/m \to \mathbb{Q}(\omega)$ $\rightsquigarrow$ $H_2^{\varphi}(X;\mathbb{Q}(\omega))$, with a hoisted intersection form $Q_{X_m}^{\varphi}$

Define: $\sigma^{\varphi}(X) := \sigma(Q_{X_m}^{\varphi})$

Fact: this is Hermitian

[Can use this to get invariants of 3-mflds]

$Y^3$ closed, oriented 3-mfld.

$\chi : \pi_1(Y) \to \mathbb{Z}/m$ , $m \geq 1$

fact abt bordism gps: $\exists W^4$ compact, oriented , connected
& $\varphi : \pi_1 W \to \mathbb{Z}/m$
& $r \geq 0$ s.t. $\partial W = rY$

& $$\begin{tikzcd} \pi_1(W) \arrow[r, "\varphi"] & \mathbb{Z}/m \\ \pi_1(Y_i) \arrow[u, "i_*"] \arrow[ur, "\chi"'] & \end{tikzcd}$$

Then define the <u>Casson-Gordon signature</u> of $(Y,\chi)$ by $\sigma_{CG}(Y,\chi) := \frac{1}{r}\left(\sigma^{\varphi}(W) - \sigma(W)\right)$.

Sketch proof that $\sigma_{CG}(Y,\chi)$ is well-defined

fact: $\Omega_4(\mathbb{Z}/m)$ generated by $\mathbb{CP}^2$

$\sigma$ is a bordism invt (also twisted)

& $\mathbb{CP}^2$ simply connected

so $\sigma^{\varphi}(W) - \sigma(W) = 0$ for closed $W$. $\square$

[Computations next week]